Sound generation in the presence of moving surfaces with application to internally generated aircraft engine noise

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In many cases of technological interest solid boundaries play a direct role in the aerodynamic sound generation process and their presence often results in a large increase in the acoustic radiation. A generalized treatment of the emission of sound from moving boundaries is presented. The approach is similar to that of Ffowcs Williams and Hawkings (1969) but the effect of the surrounding mean flow is explicitly accounted for. The results are used to develop a rational framework for the prediction of internally generated aero-engine noise. The final formulas suggest some new noise sources that may be of practical significance.

1. Introduction

Aeroacoustics was put on a rational basis when Lighthill introduced his acoustic analogy equation in 1952. Ffowcs Williams and Hawkings (1969) extended this result to account for the effects of solid boundaries in arbitrary motion. More modern approaches to the aerodynamic sound problem are based on the linearized inhomogeneous Euler (LIE) equations with the nonlinearity lumped into the inhomogeneous term which, as in the original Lighthill approach, are treated as known source terms (Bailly, Lafon, and Candel 1995; Bogey, Bailly, and Juvé 2002; Goldstein 1999, 2000, 2002). Goldstein (2002) showed that the full Navier-Stokes equations can always be recast into the form of the linearized Navier-Stokes (LNS) equations but with the viscous stress perturbation replaced by a certain generalized Reynolds stress and the heat flux perturbation replaced by a generalized stagnation enthalpy flux. . The primary purpose of this paper is to use this general result to derive an extension of the Ffowcs Williams-Hawkins (1969) equation that accounts for the effects of a non-uniform mean flow field The resulting general formula is then used to develop a sequence of progressively more complex equations for predicting the internally generated noise from modern jet engines-with each successive equation requiring less modeling than its predecessor. Some potentially important new noise sources are also identified.

2. The fundamental equation

Goldstein (2002) showed that the Navier-Stokes equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}\rho v_j=0, \tag{2.1}$$

$$\frac{\partial}{\partial t}\rho v_i+\frac{\partial}{\partial x_j}\rho v_i v_j+\frac{\partial p}{\partial x_i}=\frac{\partial}{\partial x_j}\sigma_{ij}, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho h_o-p)+\frac{\partial}{\partial x_j}\rho v_j h_o=-\frac{\partial q_j}{\partial x_j}+\frac{\partial}{\partial x_j}v_i\sigma_{ij} \tag{2.3}$$

where

$$h_o\equiv h+\frac{1}{2}v^2 \tag{2.4}$$

denotes the stagnation enthalpy, h denotes the enthalpy, t denotes the time, $x \equiv \{x_1,x_2,x_3\}$ are Cartesian coordinates, p denotes the pressure, ρ denotes the density, $v=\{v_1,v_2,v_3\}$ is the fluid velocity, σ_{ij} is the viscous stress tensor, q_i is the heat flux vector and the dependent variables satisfy the ideal gas law:

$$p=\rho RT, \quad h=c_pT, \tag{2.5}$$

with $R=c_p-c_v$ being the gas constant, c_p and c_v are the specific heats at constant pressure and volume and T the absolute temperature, can be recast into the form of the linearized Navier-Stokes equations by dividing the dependent variables

$$\rho=\overline{\rho}+\rho', \quad p=\overline{p}+p', \quad h=\widetilde{h}+h', \quad v_i=\widetilde{v}_i+v_i', \tag{2.6}$$

as well as the viscous terms σ_{ij} and q_i, into their 'base flow' components $\overline{\rho},\overline{p},\widetilde{h},\widetilde{v}_i,\widetilde{\sigma}_{ij}$, and $\widetilde{q}_i$, and into their 'residual' components $\rho', p', h', v_i', \sigma_{ij}'$, and q_i', and requiring that the former satisfy the inhomogeneous Navier–Stokes equations

$$\frac{D_o}{Dt}\,\bar{\rho} = 0, \tag{2.7}$$

$$D_o\,\bar{\rho}\tilde{v}_i + \frac{\partial \bar{p}}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\tilde{T}_{ij} + \tilde{\sigma}_{ij}\right) + \bar{f}_i, \tag{2.8}$$

$$D_o\bar{\rho}\tilde{h}_o - \frac{\partial \bar{p}}{\partial t} = \frac{D_o \tilde{H}_o}{Dt} + \frac{\partial}{\partial x_j}\left(\tilde{H}_j - \tilde{q}_j + \tilde{v}_i\tilde{\sigma}_{ij}\right) + \bar{f}_4 + \tilde{v}_i\bar{f}_i, \tag{2.9}$$

along with an ideal gas law equation of state,

$$\tilde{h} = c_p\tilde{T} = \frac{c_p}{R}\frac{\bar{p}}{\bar{\rho}}, \tag{2.10}$$

where the operator D_o/Dt is *not* the usual convective derivative but is defined by

$$\frac{D_o}{Dt} f \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{v}_j f\right) \tag{2.11}$$

for any function f,

$$\tilde{h}_o \equiv \tilde{h} + \frac{1}{2}\tilde{v}^2 \tag{2.12}$$

is the base flow stagnation enthalpy, and the 'sources strengths' $\tilde{T}_{ij}$, $\tilde{H}_o$, and $\tilde{H}_j$, and 'sources' $\bar{f}_i, \bar{f}_4$, which are assumed to be localized, can be arbitrarily specified.

The LNS equations, which are the governing equations for residual variables, are given by

$$\frac{D_o}{Dt}\rho' + \frac{\partial}{\partial x_j}\bar{\rho}u_i' = 0, \tag{2.13}$$

$$\frac{D_o}{Dt}\bar{\rho}u_i' + \bar{\rho}u_j'\frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial}{\partial x_i}p_o' - \frac{\rho'}{\bar{\rho}}\frac{\partial}{\partial x_j}\tilde{\tau}_{ij} = \frac{\partial}{\partial x_j}\left(e_{ij}' - \tilde{e}_{ij}\right) - \bar{f}_i, \tag{2.14}$$

and

$$\frac{1}{\gamma-1}\left(\frac{D_o}{Dt}p_o' + \gamma\frac{\partial}{\partial x_j}\bar{p}u_o'\right) + p_o'\frac{\partial \tilde{v}_j}{\partial x_j} - u_i'\frac{\partial \tilde{\tau}_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(e_{4j}' - \tilde{e}_{4j}\right) + \left(e_{ij}' - \tilde{e}_{ij}\right)\frac{\partial \tilde{v}_i}{\partial x_j} - \bar{f}_4, \tag{2.15}$$

where $\gamma \equiv c_p/c_v$ is the specific heat ratio, the dependent variables p_o' and u_i' are defined by

$$p_o' \equiv p' + \frac{\gamma-1}{2}\rho v'^2 + (\gamma-1)\tilde{H}_o, \tag{2.16}$$

$$u_i' \equiv \frac{u_i}{\bar{\rho}} = \rho\frac{v_i'}{\bar{\rho}}, \tag{2.17}$$

the source strengths $e_{ij}' - \tilde{e}_{ij}$ and $e_{4j}' - \tilde{e}_{4j}$ are defined by

$$e_{ij}' \equiv -\rho v_i' v_j' + \frac{\gamma-1}{2}\delta_{ij}\rho v'^2 + \sigma_{ij}', \tag{2.18}$$

$$e_{4i}' \equiv -\rho v_i' h_o' - q_i' + \sigma_{ij} v_j', \tag{2.19}$$

$$\tilde{e}_{ij} \equiv \tilde{T}_{ij} - \delta_{ij}(\gamma-1)\tilde{H}_o, \tag{2.20}$$

$$\tilde{e}_{4i} \equiv \tilde{H}_i - \tilde{T}_{ij}\tilde{v}_j, \tag{2.21}$$

and we have put

$$h'_o = h' + \frac{1}{2}v'^2; \quad \widetilde{\tau}_{ij} \equiv \delta_{ij}\overline{p} - \widetilde{T}_{ij} - \widetilde{\sigma}_{ij}. \tag{2.22}$$

The five LNS equations (2.13) to (2.15) can be written more compactly by introducing the five dimensional operator

$$\mathcal{L}_{\mu\nu} \equiv \delta_{\mu\nu}\frac{D_o}{Dt} + (\gamma-1)\delta_{\nu 4}\partial_\mu + \partial_\nu\left(\frac{\widetilde{c^2}}{\gamma-1}\delta_{\mu 4} + \delta_{\mu 5}\right) + K_{\mu\nu} \tag{2.23}$$

where μ, ν =1,2,…,5, while the Latin indices i, j are restricted to the range 1, 2, 3.

$$K_{\mu\nu} \equiv \partial_\nu \widetilde{v}_\mu - \frac{1}{\overline{\rho}}\frac{\partial\widetilde{\tau}_{\mu j}}{\partial x_j}\delta_{\nu 5} + \left((\gamma-1)\frac{\partial\widetilde{v}_j}{\partial x_j}\delta_{\nu 4} - \frac{1}{\overline{\rho}}\frac{\partial\widetilde{\tau}_{\nu j}}{\partial x_j}\right)\delta_{\mu 4} \tag{2.24}$$

$$\partial\mu \equiv \begin{cases} \dfrac{\partial}{\partial x_i}, & i = \mu = 1,2,3 \\ 0, & \text{otherwise} \end{cases} \tag{2.25}$$

$$\widetilde{v}_\mu, \ \frac{\partial\widetilde{\tau}_{\mu j}}{\partial xj} \equiv \begin{cases} \widetilde{v}_i, \ \dfrac{\partial\widetilde{\tau}_{ij}}{\partial x_j}, & i = \mu = 1,2,3 \\ 0, & \text{otherwise} \end{cases}, \tag{2.26}$$

$$\widetilde{c^2} = \gamma\overline{p}/\overline{\rho} \tag{2.27}$$

is the base flow speed of sound, the Einstein summation convention is still being used and $\delta_{\mu\nu}$ is the five dimensional Kronecker delta.

Equations (2.13) to (2.15) now become

$$\mathcal{L}_{\mu\nu}\, u_\nu \;=\; s_\mu \tag{2.28}$$

where

$$\{u_\mu\} \equiv \left\{\bar{\rho} u_i',\; \frac{p_o'}{\gamma - 1},\; \rho'\right\} \tag{2.29}$$

is the five dimensional dependent variable vector and the five dimensional source vector s_μ is given by

$$s_\mu \equiv \frac{\partial}{\partial x_j}\left(e_{\mu j}' - \tilde{e}_{\mu j}\right) + \delta_{\mu 4}\left(e_{ij}' - \tilde{e}_{ij}\right)\frac{\partial \tilde{v}_i}{\partial x_j} \text{ for } \mu = 1,\ldots 4 \tag{2.30}$$

The vector Greens function $g_{\nu\sigma} = (x, t | x_o, t_o)$ for $\mathcal{L}_{\mu\nu}$ is defined by (Morse and Feshbach, 1953).

$$\mathcal{L}_{\mu\nu}\, g_{\nu\sigma} \;=\; \delta_{\mu\sigma}\,\delta(\mathbf{x} - \mathbf{x}_o)\delta(t - t_o). \tag{2.31}$$

Its adjoint $g_{\nu\sigma}^a = (x, t | x_o, t_o)$ satisfies (Morse and Feshbach,1953 and Tam and Auriault, 1998)

$$\mathcal{L}_{\mu\nu}^a\, g_{\nu\sigma}^a \;=\; \delta_{\mu\sigma}\,\delta(\mathbf{x} - \mathbf{x}_o)\delta(t - t_o) \tag{2.32}$$

where

$$\mathcal{L}^a_{\mu\nu} \equiv -\delta_{\nu\mu}\frac{\overline{D}}{Dt} - (\gamma-1)\,\delta_{\mu 4}\partial_\nu - \left(\frac{\widetilde{c^2}}{\gamma-1}\delta_{\nu 4} + \delta_{\nu 5}\right)\partial_\mu + K_{\nu\mu} \tag{2.33}$$

and

$$\frac{\overline{D}}{Dt} \equiv \frac{\partial}{\partial t} + \widetilde{v}_i\frac{\partial}{\partial x_i} \tag{2.34}$$

denotes the ordinary convective derivative. It is easy to show from this that

$$v_\mu \mathcal{L}_{\mu\nu} u_\nu - u_\nu \mathcal{L}^a_{\nu\mu} v_\mu = \frac{D_o}{Dt} v_\mu u_\mu + \frac{\partial}{\partial x_i}\left[(\gamma-1) v_i u_4 + \left(\frac{\widetilde{c^2}}{\gamma-1} v_4 + v_5\right) u_i\right] \tag{2.35}$$

for any two five dimensional vectors $\{v_\mu\}$ and $\{u_\nu\}$.

We expect $g_{\nu\sigma}(\boldsymbol{x},t|\boldsymbol{x}_o,t_o)$ to vanish as $t \to \infty$ for all finite $\boldsymbol{x}$ when the base flow (2.7) to (2.9) is globally stable and causality should insure that it vanishes as $t \to -\infty$. We, therefore, require that

$$\lim_{t\to\pm\infty} \left\{g_{\nu\sigma}(\boldsymbol{x},t|\boldsymbol{x}_o,t_o),\ g^a_{\nu\sigma}(\boldsymbol{x},t|\boldsymbol{x}_o,t_o)\right\} \to 0 \tag{2.36}$$

since it is reasonable to suppose that $g^a_{\nu\sigma}$ will exhibit similar behaviour.

Now let $v(t)$ denote an arbitrary region of space bounded by the (in general moving) surface $S(t)$ and let $\boldsymbol{A} = \{A_i\}$ be an arbitrary vector and $\Phi(\boldsymbol{x},t)$ an arbitrary function defined on $v(t)$. Then the divergence theorem shows that

$$\int_{S(t)} A_i \hat{n}_i \, dS(\boldsymbol{x}) = \int_{v(t)} \frac{\partial A_i}{\partial x_i} d\boldsymbol{x} \tag{2.37}$$

and the three-dimensional Leibniz's rule (Goldstein 1976) shows that

$$\frac{d}{dt}\int_{v(t)} \Phi\, dx = \int_{v(t)} \frac{\partial \Phi}{\partial t} dx + \int_{S(t)} V_i^s\, \hat{n}_i\, \Phi\, dS(x) \tag{2.38}$$

where $\hat{\boldsymbol{n}} = \{\hat{n}_i\}$ is the unit outward drawn normal to S and $V^s = \{V_i^s\}$ is the corresponding surface velocity at any point $\boldsymbol{x}$ of S.

Setting v_μ equal to $g^a_{\mu\kappa}(\boldsymbol{x}, t|\boldsymbol{x}_o, t_o)$ in (2.35), letting u_ν be the solutions to (2.28), using (2.36) to (2.38) and omitting terms that are negligibly small as T goes to plus or minus infinity, shows that

$$u_\kappa(\boldsymbol{x}_o, t_o) = \int_{-T}^{T} \int_{v(t)} g^a_{\mu\kappa}(\boldsymbol{x}, t|\boldsymbol{x}_o, t_o)\, s_\mu(\boldsymbol{x}, t)\, d\boldsymbol{x} dt$$

$$-\int_{-T}^{T} \int_{S(t)} \hat{n}_i \left[\left(\tilde{v}_i - V_i^s\right) g^a_{\mu\kappa} u_\mu + g^a_{i\kappa} u_4 (\gamma - 1) + u_i \left(\frac{\widetilde{c}^2}{\gamma - 1} g^a_{4\kappa} + g^a_{5\kappa} \right) \right] dS dt \tag{2.39}$$

where T denotes a very large but finite time interval. This formula expresses the solution to equation (2.28) in terms of the volume source distribution s_μ and the values of $\{u_\mu\}$ on some arbitrarily moving surface $S(t)$. However, the adjoint vector Greens function $g^a_{\mu\kappa}$ is not uniquely determined by (2.32). and can be required to satisfy certain boundary conditions on a portion, say $S_1(t)$, of the surface $S(t)$ that encloses a region, say $v_1(t)$, that contains $v(t)$ (see fig. 1).

We could also have chosen $u_\kappa(x,t)$ in (2.39) to be the Greens function $g_{\kappa\lambda}(\boldsymbol{x}, t|\boldsymbol{x}_1, t_1)$. In which case s_μ would denote $\delta_{\mu\kappa}\, \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1)$ and (2.39) would show that $g_{\mu\nu}$ and $g^a_{\mu\kappa}$ satisfy the reciprocity relation

$$g_{\kappa\lambda}(\boldsymbol{x}_o, t_o|\boldsymbol{x}_1, t_1) = g^a_{\lambda\kappa}(\boldsymbol{x}_1, t_1|\boldsymbol{x}_o, t_o) \tag{2.40}$$

if

$$\hat{n}_i\left(\tilde{v}_i - V_i^s\right) = 0 \tag{2.41}$$

on $S_1(t)$ and $g_{\mu\nu}$ and $g^a_{\mu\kappa}$ satisfy the impedance boundary conditions

$$\left(\frac{\widetilde{c^2}}{\gamma-1} g^a_{4\kappa} + g^a_{5\kappa}\right) = -Z(\boldsymbol{x},t)\hat{n}_i g^a_{i\kappa} \tag{2.42}$$

and

$$(\gamma-1)g_{4\kappa} = \hat{n}_i g_{i\lambda} Z(\boldsymbol{x},t) \tag{2.43}$$

for all $\boldsymbol{x}$ on $S_1(t)$ and $\boldsymbol{x}_o$ in $v_1(t)$. This would almost certainly be the case if S_1 were a large stationary sphere whose radius approaches infinity (see fig. 1). In which case we would refer to $g_{\mu k}$ and $g^a_{\mu\kappa}$ as free space Green's functions.

Again interpreting u_k to be a solution to (2.28) (which we shall do for the remainder of the paper), inserting (2.30) into (2.39) and using the divergence theorem (2.37) to transfer the spatial derivatives to the Greens function shows that

$$u_\kappa(\boldsymbol{x}_o,t_o) = \int_{-T}^{T}\int_{v(t)} \gamma^a_{\nu j,\kappa}\left(e'_{\nu j} - \tilde{e}_{\nu j}\right)d\boldsymbol{x}dt - \int_{-T}^{T}\int_{S(t)} \hat{n}_i I^\kappa_j \, dSdt - F_\kappa \tag{2.44}$$

where we have put

$$\gamma^a_{\nu j,\kappa}\left(\boldsymbol{x}_o,t_o \middle| \boldsymbol{x},t\right) \equiv \frac{\partial}{\partial x_j} g^a_{\nu\kappa} + \frac{\partial \tilde{v}_\nu}{\partial x_j} g^a_{4\kappa} \tag{2.45}$$

$$I^\kappa_j \equiv \left(\tilde{v}_j - V^s_j\right)g^a_{\mu\kappa}u_\mu + (\gamma-1)u_4 g^a_{j\kappa} + u_j\left(\frac{\widetilde{c^2}}{\gamma-1} g^a_{4\kappa} + g^a_{5\kappa}\right) - \left(e'_{\nu j}v - \tilde{e}_{\nu j}v\right)g^a_{\nu\kappa} \tag{2.46}$$

and set

$$F_\kappa = \int_{-T}^{T} \int_{v(t)} g_{\mu\kappa}^a \bar{f}_\mu dx dt \tag{2.47}$$

Inserting the definitions (2.6) and (2.16) to (2.21) into (2.46) shows that

$$I_j^\kappa \equiv \left(v_j - V_j^s\right) g_{\mu\kappa}^a u_\mu - \left(\rho v_i' v_j' + e_{ij}' - \tilde{e}_{ij}\right) g_{i\kappa}^a + \left(p' + \frac{\gamma - 1}{2}\rho v'^2 + (\gamma - 1)\tilde{H}_o\right) g_{j\kappa}^a$$

$$+ \frac{g_{4\kappa}^a}{\gamma - 1}\left[\widetilde{c2}\,\rho v_j' - v_j'\left(p' + \frac{\gamma - 1}{2}\rho v'^2 + (\gamma - 1)\tilde{H}_o\right) - (\gamma - 1)\left(e_{4j}' - \tilde{e}_{4j}\right)\right] + \bar{\rho} v_j' \, g_{5\kappa}^a \tag{2.48}$$

which can be further rearranged to obtain

$$I_j^\kappa \equiv \left(\tilde{v}_j - V_j^s\right) g_{\mu\kappa}^a u_\mu + \left(p' \delta_{ij} - \sigma_{ij}' + \tilde{T}_{ij}\right) g_{i\kappa}^a$$

$$+ \left\{\frac{\bar{\rho}\,\widetilde{c2}}{\gamma - 1} v_j' + \left[\delta_{j\kappa}\left(p' - \tilde{H}_o\right) - \sigma_{j\kappa}\right] v_\kappa' + q_j' + \tilde{e}_{4j}\right\} g_{4\kappa}^a + \bar{\rho} v_j' \, g_{5\kappa}^a \tag{2.49}$$

We now require that the base flow velocity satisfy the slip boundary conditions (2.41) on the portion S_1 of S where $g_{\nu\kappa}^a$ satisfies the boundary condition (2.42) and consider only those flows that satisfy the impedance boundary condition

$$p' = \frac{\rho}{\bar{\rho}} Z \hat{n}_i v_i' \tag{2.50}$$

on this surface. The remaining portion, say $S_o(t)$, of $S(t)$ will, in most cases, be a solid surface, which means that

$$\hat{n}_j\left(v_j - V_j^s\right) = 0 \tag{2.51}$$

there. It is worth noting that the boundary conditions (2.41) and (2.52) are appropriate even when the flow is inviscid. It now follows from (2.18) to (2.20), (2.41), (2.42), (2.46), and (2.49) to (2.52) that equation (2.44) becomes

$$u_K(x_o,t_o) = -\int_{-T}^{T} \int_{v(t)} \gamma^a_{vj,K}(x_o,t_o|x,t)\left[e'_{vj}(x,t) - \tilde{e}_{vj}(x,t)\right] dx dt$$

$$-\int_{-T}^{T} \int_{S(t)} g^a_{vK}(x_o,t_o|x,t)\left[\varepsilon'_{vj}(x,t) - \tilde{\varepsilon}_{vj}(x,t)\right]\hat{n}_j \, dS(x) dt - F_K(x_o,t_o) \qquad (2.52)$$

where the five dimensional surface stresses are defined by

$$\varepsilon'_{ij} = \begin{cases} -\rho v'_i v'_j + \sigma'_{ij} & \text{on } S_1 \\ -p'\delta_{ij} + \sigma'_{ij} & \text{on } S_o \end{cases} \qquad (2.53)$$

$$\varepsilon'_{4j} = \begin{cases} e'_{4j} \equiv -\rho v'_j h'_o - q'_j + \sigma_{ji} v'_i & \text{on } S_1 \\ -v'_j\left(\dfrac{\bar{\rho}\widetilde{c^2}}{\gamma - 1} + p' - \tilde{H}_o\right) - q'_j + \sigma_{ji} v'_i & \text{on } S_o \end{cases} \qquad (2.54)$$

$$\varepsilon'_{5j} = \begin{cases} 0 & \text{on } S_1 \\ -\bar{\rho} v'_j & \text{on } S_o \end{cases} \qquad (2.55)$$

$$\tilde{\varepsilon}_{vj} = \begin{cases} \tilde{T}_{ij} & v = i = 1,2,3. \\ \tilde{e}_{4j} & v = 4 \\ 0 & v = 5 \end{cases} \qquad (2.56)$$

This exact equation is the main result of the paper. It can also be rewritten in terms of the direct Greens function $g_{v\mu}$ by using the reciprocity relation (2.40), which must then satisfy the impedance boundary condition (2.43). It is usually only necessary to consider the fourth component of this equation, since aeroacoustics is primarily concerned with predicting pressure fluctuations.

Notice that the inviscid component of the normal surface stress $\varepsilon'_{\nu j}\hat{n}_j$ will vanish on S_1 when this surface is stationary and impermeable. In the more general case, it demonstrates that acoustic liners can, by their very presence, introduce new sources of sound. The first three components of $\varepsilon'_{\nu j}\hat{n}_j$ represent a normal surface force on S_o and correspond to what is usually referred to as a surface dipole (see discussion at the end of §3.1 below).

The first two terms in the fourth component can be rewritten as

$$-v_j' \frac{\bar{\rho}\tilde{c}^2}{\gamma-1}\left(1+\frac{p'}{\bar{p}}(\gamma-1)\right)$$

and it follows from (2.52) that

$$\hat{n}_i v_i' \equiv \hat{n}_i\left(v_i - \tilde{v}_i\right) = \hat{n}_i\left(V_i^S - \tilde{v}_i\right) = V_n' \tag{2.57}$$

is the normal component of the surface velocity relative to the base flow velocity. Then since $p'(\gamma-1)/\bar{p}$ is likely to be small compared to unity, these two terms plus the fourth component of $\varepsilon'_{\nu j}\hat{n}_j$ are probably best interpreted as a surface displacement source.

The volume source $e'_{\nu j} - \tilde{e}_{\nu j}$ is basically of the quadrupole type with the first three components corresponding to the fluctuating Reynolds stress and the fourth component corresponding to a fluctuating stagnation enthalpy flux. Gliebe and Mani (1999, p. 3) recently argued that the volume quadrupole is likely to be the dominant source of turbulence blade row interaction noise. This would suggest that the fluctuating enthalpy flux should play an important role in the generations of turbine noise.

3. Application to internally generated aero-engine noise

The most important application of (2.53) is to the prediction of the noise generated inside a turbojet engine. In which case, it would be appropriate to associate the finite part of S_1 with the surface of the engine nacelle and perhaps the internal (axial) boundary of the flow path. The remaining portion of S_o of S would most appropriately be associated with the (fixed and rotating) blade surfaces, for fan, compressor, and/or turbine noise prediction studies.

3.1 *The infinite duct model*

The simplest fan, compressor, and/or turbine noise models assume that the (rotating and stationary) blade rows are embedded in an infinite straight duct (either circular or annular) in which there is a parallel (i.e., uni-directional transversely sheared) mean flow (Goldstein, 1976; Gliebe and Mani 1999). This corresponds to setting the arbitrary source strengths $\tilde{T}_{ij}$, $\tilde{H}_o$, and $\tilde{H}_j$ and source terms $\bar{f}_v$ to zero in (2.7) to (2.9) which then reduce to the usual Euler equations, which in turn posses the steady solution

$$U_i = \delta_{i1} U(x_2, x_3), \quad \bar{p} = \text{ const.}, \quad \bar{\rho} = \rho(x_2, x_3) \tag{3.1}$$

that satisfies the normal velocity boundary condition (2.41) on the duct walls.

It can be verified by direct substitution that the $\sigma = 4$ component of (2.32) can be expressed in terms of a single scalar Greens function $G^a(\boldsymbol{x}, t|\boldsymbol{x}_o, t_o)$ by

$$(\gamma - 1) g_{i4}^a = -\tilde{c}^2 \left(\frac{\partial}{\partial x_i} \frac{\bar{D} G^a}{Dt} + 2 \frac{\partial U}{\partial x_i} \frac{\partial G^a}{\partial x_1} \right) \tag{3.2}$$

$$g_{44}^a = \frac{\bar{D}^2 G^a}{Dt^2} \tag{3.3}$$

$$g_{54}^a = 0 \tag{3.4}$$

where G^a satisfies the adjoint wave equation

$$L^a G^a = \delta(\boldsymbol{x} - \boldsymbol{x}_o) \delta(t - t_o)$$

with (see Tam and Auriault 1998)

$$L^a \equiv \frac{\overline{D}^3}{Dt^3} - \frac{\partial}{\partial x_i} \tilde{c}^2 \frac{\partial}{\partial x_i} \frac{\overline{D}}{Dt} - 2 \frac{\partial}{\partial x_j} \tilde{c}^2 \frac{\partial U}{\partial x_j} \frac{\partial}{\partial x_1} \tag{3.5}$$

along with the impedance boundary conditions (2.42) on the duct surface. To avoid getting into a discussion of the liner generated sound, we consider only the hard walled case $Z \to \infty$. Then the fourth component of (2.53) becomes

$$p'_o = -\int_{-T}^{T} \int_{V} (\gamma - 1)\, \gamma^a_{vj,4}\, e'_{vj}\, dx\, dt + \int_{-T}^{T'} \int_{S_o(t)} \left[(\gamma-1) g^a_{i4}\, \hat{n}_i\, p' + g^a_{44}\, \overline{\rho}\, \tilde{c}^2\, V'_n \left(1 + \frac{p'}{\overline{p}} (\gamma - 1) \right) \right] dS_o\, dt \tag{3.6}$$

where

$$(\gamma-1)\gamma^a_{ij,4} = -\frac{\partial}{\partial x_j} \tilde{c}^2 \left(\frac{\partial}{\partial x_i} \frac{\overline{D} G^a}{Dt} + 2 \frac{\partial U}{\partial x_i} \frac{\partial G^a}{\partial x_1} \right) + \delta_{i1} \frac{\partial U}{\partial x_j} (\gamma - 1) \frac{\overline{D}^2 G^a}{Dt} \tag{3.7}$$

and

$$\gamma^a_{4j,4} = \frac{\partial}{\partial x_j} \frac{\overline{D}^2 G^a}{Dt^2} \tag{3.8}$$

Most turbomachinery analyses assume that the mean flow and sound speed are uniform (Goldstein 1976; Gliebe and Mani 1999). In which case (3.6) to (3.8) become

$$\frac{p_o'}{\widetilde{c^2}} = \int_{-T}^{T} \int_{V_1} \left[\frac{\partial^2 G^*}{\partial x_i \partial x_j} e_{ij}' + \frac{\partial}{\partial x_j} \left(\frac{1}{\widetilde{c^2}} \frac{\bar{D} G^*}{Dt} \right) e_{4j}' \right] dx dt$$

$$- \int_{-T}^{T} \int_{S_o(t)} \left[\hat{n}_i \frac{\partial G^*}{\partial x_i} p' - \bar{\rho} V_n' \left(1 + \frac{p'}{\bar{p}} (\gamma - 1) \frac{\bar{D} G^*}{Dt} \right) \right] dS_o dt \qquad (3.9)$$

where

$$G^* \equiv \frac{\bar{D} G^a}{Dt} \qquad (3.10)$$

satisfies a second order wave equation.

This should be compared with equation (4.12) of Goldstein (1976) which is derived from Lighthill's result. The only difference is that the volume quadrupole source $\delta_{ij} \left(p' - \widetilde{c^2} \rho' \right) \frac{\partial^2 G^*}{\partial x_i \partial x_j}$ in the latter is replaced by the volume dipole source $\frac{\partial}{\partial x_j} \frac{1}{\widetilde{c^2}} \frac{\bar{D} G^*}{Dt} e_{4j}'$ in the former, which induces an additional factor in the volume displacement surface source. This difference is due to the fact that the basic differential equations (2.13) to (2.15) on which this analysis is based, do not lead to Lighthill's equation in the limit when the base flow and sound speed become constant, but they do, with some slight rearrangement, reduce to a version of Lighthill's equation originally derived by Lilley (1974). Lighthill originally suggested that the inconvenient $\delta_{ij} \left(p' - \widetilde{c^2} \rho' \right)$ term that appeared in his stress tensor should be associated with non-isentropic density fluctuations, which are presumably produced by viscous effects and are therefore likely to be negligible. However, Lilley (1974) pointed out that this term also has a component associated with isentropic density fluctuations, which can be important when the mean density (i.e., temperature) is non-constant, and proceeded to derive an equation which separates out the two components. Unfortunately, the resulting source term contained a dipole component which corresponds to the dipole source term that appears in the present formulation. Lilley (1996) applied his result to the prediction of jet noise. He modeled the e_{4j}' source by using the strong Reynolds analogy—though he did not call it that (see Smits and Dussauge 1996, pp.

125, 130) —and was able to obtain a good agreement with the hot jet data available at that time—even though the strong Reynolds analogy, which is quite accurate for boundary layer-type flows, does not seem to work very well for jets and free shear layers (Smits and Dussauge 1996). This term should as noted above play an important role in the prediction of turbine noise. Finally it is worth noting that the present results justify the source interpretation given at the end of §2.

3.2 More general steady base flows

As noted by Gliebe and Mani (1999, p. 63), a major difficultly with the parallel flow model is that it cannot account for the large turning of the mean flow that occurs downstream of the blade rows. This difficulty, as well as a number of others, can be avoided by taking the base flow to be the actual mean flow field through the engine. This choice of base flow has the additional benefit of allowing the observation point in (2.53) to be moved into the far-field, where one is actually interested in calculating the sound.

The over bar on the dependent base flow variables then denotes the time average

$$\bar{\bullet} \equiv \lim_{T\to\infty} \int_{-T}^{T} \bullet(\boldsymbol{x},t)\,\mathrm{d}\,t, \tag{3.11}$$

where the dot is a place holder for ρ, v_i, p, and h, which must be set to zero during the time that the blade passes through the point $\boldsymbol{x}$ when averaging within the blade passages of a rotating blade row.

$$\tilde{\bullet} \equiv \left(\overline{\rho\bullet}/\bar{\rho}\right) \tag{3.12}$$

denotes a Favre averaged quantity (Lele 1994) for all variables except $\tilde{h}_o$, which is defined by (2.12). Notice that equation (2.10) is completely consistent with the ideal gas law $p = \rho RT$ when the tilde is defined in this fashion.

The time derivatives now drop out of the base flow equations (2.7) to (2.9), which do not, of course, form a closed system. The source strengths $\tilde{T}_{ij}$, $\tilde{H}_o$, and $\tilde{H}_j$ are given by

$$\tilde{T}_{ij} = -\bar{\rho}\left(\widetilde{v_i v_j} - \tilde{v}_i\tilde{v}_j\right), \tag{3.13}$$

$$\tilde{H}_o = \frac{1}{2}\bar{\rho}\left(\widetilde{v^2} - \tilde{v}^2\right) = \frac{1}{2}\tilde{T}_{ij}, \tag{3.14}$$

$$\tilde{H}_j = -\bar{\rho}\left(\widetilde{h_o v_j} - \tilde{h}_o \tilde{v}_j\right) - \tilde{H}_o \tilde{v}_j, \tag{3.15}$$

which can be written more compactly as

$$\tilde{T}_{ij} = -\bar{\rho}\widetilde{v_i' v_j'}, \tag{3.16}$$

$$\tilde{H}_o = \frac{1}{2}\bar{\rho}\widetilde{(v')^2}, \tag{3.17}$$

and

$$\tilde{e}_{4i} \equiv \tilde{H}_i - \tilde{T}_{ij}\tilde{v}_{ij} = -\bar{\rho}\widetilde{h_o' v_i'} \tag{3.18}$$

The dipole volume source $\bar{f}_\mu$ are given by (see appendix A)

$$\bar{f}_i = -\lim_{N\to\infty}\frac{\Omega}{2N}\sum_{j=-N}^{N}\frac{f_i\left(x, t_s(x) + j/\Omega\right)}{V_n\left(x, t_s(x) + j/\Omega\right)} \tag{3.19}$$

and

$$\bar{f}_4 = -\lim_{N\to\infty}\frac{\Omega}{2N}\sum_{j=-N}^{N}\frac{v_i' f_i\left(x, t_s(x) + j/\Omega\right) + q\left(x, t_s(x) + j/\Omega\right)}{V_n\left(x, t_s(x) + j/\Omega\right)} \tag{3.20}$$

where Ω is the angular rotational frequency of the turbomachine $t_s(x)$ is the time when the blade surface has reached the point x (measured from some fixed reference time), V_n is the normal surface velocity,

$$f_i \equiv \hat{n}_i \left(p\delta_{ij} - \sigma_{ij} \right) \tag{3.21}$$

is the normal blade force and

$$q = \hat{n}_i q_i \tag{3.22}$$

is the normal heat flux.

The base flow equations arc now the ordinary RANS equations with additional dipole source terms. These equations are, of course, not closed and require the introduction of some sort of model relating the source terms to the mean flow variables $\tilde{v}_i$, $\bar{\rho}$, $\bar{p}$, and $\tilde{h}_o$ and/or their derivative. The source strengths (3.16) to (3.18) consist of a deterministic component due to the periodic motion of the blades and a random component, which is usually modeled by the Boussinesq approximation (Speziale 1991; Speziale and So 1998)

$$\tilde{T}_{ij} = \mu_T \left(\frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \tilde{v}_k}{\partial x_k} \delta_{ij} \right) + \frac{2}{3} \tilde{H}_o \delta_{ij} \tag{3.23}$$

with a similar result for the enthalpy fluxes. The deterministic component, as well as the volume dipole sources, were first modeled by Adamczyk (1985), who also developed a computer code for obtaining an approximation to the relevant solution to the RANS equations.

3.3 Unsteady radiating base flows

The ultimate objective of internal engine aeroacoustics is to directly calculate the noise by using large scale numerical simulation (DNS). This approach may eventually succeed in predicting the periodic components of the sound field (i.e., the engine order tones) but it is unlikely to be successful in predicting the broad band component, which would require a complete knowledge of the actual turbulent flow within the engine. It is therefore important to develop an internally consistent framework that would enable the direct calculation of the pure tones and still use an acoustic analogy-type approach to determine the broad band sound. This can be accomplished by taking the base flow equations (2.7) to (2.9) to be the filtered or phase averaged Navier-Stokes equations. The over bars on the base flow variables would then denote the filtered variables

$$\bar{f} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f\left(x, t + \frac{i}{\Omega}\right) \tag{3.24}$$

where Ω is defined in the previous subsection and f can again by any of the dependent variables ρ, v_i, p, and h within the flow and should be set to zero inside the blades when averaging within a moving blade row. This filtering (or averaging) process will pass only the steady and periodic components of the motion.

The tilde is still defined by (3.12) and the base flow source terms $\tilde{T}_{ij}$, $\tilde{H}_o$, and $\tilde{H}_i$ are still given by (3.13) to (3.15) but not be (3.16) to (3.18). They account for the effect of the random turbulence on the periodic flow through the turbomachine and must again be modeled. It would probably by appropriate to still use the Boussinesq model (3.17) for this purpose since the Reynolds stresses no longer contain a deterministic component.

The pure tones can be calculated by solving these equations subject to no-slip boundary conditions on the blade surfaces $S_o(t)$ together with appropriate boundary conditions on the duct wall surfaces S_1. For simplicity, we again consider only hard walled ducts so that (2.41) still holds there.

The remaining random component of the sound field is given by the fundamental equation (2.55). However, $\hat{n}_i U_i'$ now vanishes at all points of S and it is reasonable to suppose that $\hat{n}_j \tilde{T}_{ij}$ and $\hat{n}_j \tilde{e}_{4j}$ are zero there as well—so that, in the absence of viscocity, the only remaining surface source is the blade surface dipole

$$\int_{-T}^{T'} \int_{S_o(t)} g_{i4}^{a} \hat{n}_i p' dS_o dt$$

However, the Gliebe-Mani (1999) arguments suggest that the volume source terms will also make important contributions. The adjoint Greens function must, of course, satisfy (2.42) with $Z = \infty$.

Conclusions

A generalized integral formula for predicting aerodynamic sound generation in the presence of solid surfaces and non-uniform surrounding flow has been derived. It is used to develop a set of increasingly more accurate models for predicting the internally generated aircraft engine noise—with each successive model being more computationally intensive than its predecessor. The simplest model corresponds to embedding the turbomachinery blade rows in infinite straight duct containing a uniform mean flow. The general result suggests that there are some new noise sources that have not been previously considered.

Appendix A: Derivation of volume dipole sources

Multiplying (2.2) by the Heaviside function *H(F)*, where *F(x,t)* is negative inside the blades and positive outside, integrating by parts and noting that

$$\frac{\partial F}{\partial t}+v_i\frac{\partial F}{\partial x_i}=0 \tag{A1}$$

on the moving blade surface shows that

$$\frac{\partial}{\partial t}H\rho v_i+\frac{\partial}{\partial x_j}H\rho v_i v_j+\frac{\partial}{\partial x_i}Hp=\frac{\partial}{\partial xj}H\sigma_{ij}+\delta(F)\frac{\partial F}{\partial x_j}\left(p\delta_{ij}-\sigma_{ij}\right) \tag{A2}$$

Integrating with respect to time, using (3.11) and (3.12) and comparing with (2.8) shows that

$$\bar{f}_i=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\delta(F)\frac{\partial F}{\partial x_j}\left(p\delta_{ij}-\sigma_{ij}\right)dt=\lim_{T\to\infty}\frac{1}{2T}\int_{F(x,-T)}^{F(x,T)}\delta(F)\frac{\partial F}{\partial x_j}\left(p\delta_{ij}-\sigma_{ij}\right)\frac{\partial F}{(\partial F/\partial t)}$$

$$=-\lim_{T\to\infty}\frac{1}{2\mathrm{T}}\int_{F(x,-T)}^{F(x,T)}\delta(F)\frac{f_i}{V_j^s\hat{n}_j}dF \tag{A3}$$

where we have used (2.52), (3.21), (A1) and the fact that

$$\hat{n}_i=\frac{\partial F}{\partial x_i}\bigg/\sqrt{\frac{\partial F}{\partial x_j}\frac{\partial F}{\partial x_j}} \tag{A4}$$

to obtain the last result. Integrating over the Delta function and noting that the blade surface passes each point $\boldsymbol{x}$ within the rotating blade row once per revolution yields (3.19). Equation (3.20) can be derived from (2.3) in a similar fashion.

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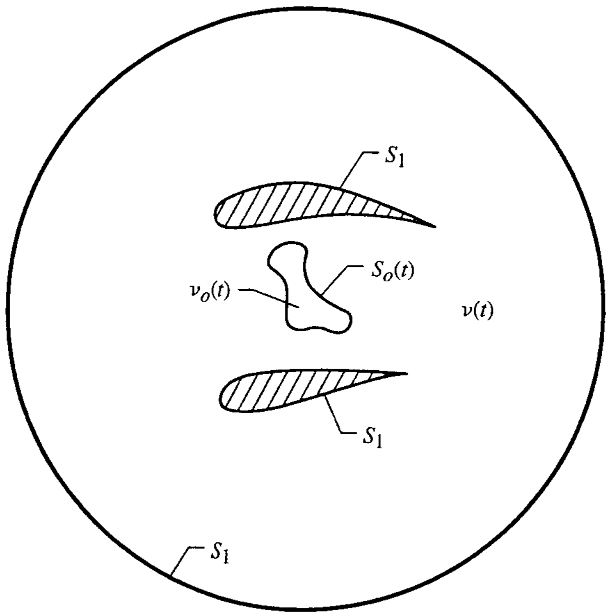

$S = S_o \; U \; S_1, \;\; \nu_1 = \nu_o \; U \; \nu$